Math 245B Lecture 22 Notes

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1 More L^p Duality and Existence of Kernel Operators

1.1 L^p duality, continued

Let's finish up our proof of L^p duality.

Theorem 1.1. *If $1 < p < \infty$, then the map $L^q \to (L^p)^*$ sending $g \mapsto \varphi_g$ is an isometric isomorphism. If μ is σ-finite, the same holds for $p = 1$.*

We are covering the case for when μ is finite. Here is a useful lemma.

Lemma 1.1. *If $\mu(X) < \infty$, then $L^p \subseteq L^1$ for all $p \geq 1$.*

Proof. By Hölder's inequality, if $f \in L^p$, then

$$\int |f| \, d\mu \int |f \mathbb{1}_X| \, d\mu \leq \|f\|_p \|\mathbb{1}_X\|_q \leq \|f\|_p (\mu(X))^{1/q}. \qquad \square$$

Last time, we showed the following proposiitions.

Proposition 1.1. $\|\varphi_g\|_{(L^p)*} = \|g\|_{L^q}$

Proposition 1.2. *If $g \in L^1$ and Σ is the set of simple functions, then*

$$\|g\|_q = \sup \left\{ \left| \int fg \, d\mu \right| : f \in \Sigma, \|f\|_p \leq 1 \right\}.$$

In particular, the left hand side is ∞ if and only if the right hand side is, as well.

Now we can complete our proof of the main theorem.

Proof. Let $\varphi \in (L^p)^*$. We proceed in steps:

Step 1: For $E \subseteq \mathcal{M}$, define $\nu(E) := \varphi(\mathbb{1}_E)$. This uses the assumption that $\mu(X) < \infty$. We claim that ν is a complex measure on $(X, \mathcal{M})$. We have $\nu(\varnothing) = \varphi(0) = 0$, and finite additivity is not too hard to check. Let's check countable additivity. Let $(E_n)_n \subseteq \mathcal{M}$ be disjoint. Then $\mathbb{1}_{\bigcup_n E_n} = \sum_n \mathbb{1}_{E_n}$. To control the tail of this series, we have

$$\left\| \sum_{n=k}^{\infty} \mathbb{1}_{E_n} \right\| = \left\| \mathbb{1}_{\bigcup_{n=k}^{\infty} E_n} \right\| = \mu \left(\bigcup_{n=k}^{\infty} E_n \right)^{1/p},$$

which goes to 0 since $\mu(X) < \infty$ and $p < \infty$. So by continuity of φ on L^p, we have

$$\nu\left(\bigcup_n E_n\right) = \varphi\left(\mathbb{1}_{\bigcup_n E_n}\right) = \varphi\left(\sum_n \mathbb{1}_{E_n}\right) = \sum_n \nu(E_n).$$

Step 2: Also, $\nu \ll \mu$. Indeed, if $\mu(E) = 0$, then $\mathbb{1}_E = 0$ μ-a.e. So $\nu(E) = 0$. By the Radon-Nikodym theorem, there exists $g \in L^1_{\mathbb{C}}(\mu)$ such that $d\nu = g\, d\mu$.

Step 3: If $f \in \Sigma$, then $\int fg\, d\mu = \int f\, d\nu = \varphi(f)$ by linearity. We know this is bounded in absolute value by $\|\varphi\|_{(L^p)^*} \|f\|_p$. Our propositions give us that $g \in L^q$ and $\|g\|_q \leq \|\varphi\|_{(L^p)^*}$. We know that that $\varphi_g|_\Sigma = \varphi|_\Sigma$. So $\varphi_g = \varphi$ on a dense subspace of L^p, so continuity gives that $\varphi_g = \varphi$. □

Corollary 1.1. *If $1 < p < \infty$, then L^p is reflexive.*

Proof. We know $1 < q < \infty$, so $(L^p)^{**} = (L^q)^* = L^p$. □

Remark 1.1. For interesting μ, L^1 and L^∞ are not reflexive.

1.2 Existence of kernel operators in L^p

Theorem 1.2. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Suppose*

1. *$K : X \times Y \to \mathbb{C}$ is measurable,*
2. *there exists $C > 0$ such that $\|(x, \cdot)\|_{L^1(\nu)} \leq C$ for μ-a.e. x,*
3. *there exists $C > 0$ such that $\|(\cdot, y)\|_{L^1(\mu)} \leq C$ for ν-a.e. y.*

Then there for all $p \in [1, \infty]$ and $f \in L^p(\nu)$, the integral

$$Tf(x) = \int_Y K(x, y) f(y)\, d\nu(y)$$

exists μ-a.e., $Tf \in L^p(\mu)$, and $\|Tf\|_{L^p(\mu)} \leq C \|f\|_{L^p(\nu)}$.

We will check the cases where $p \neq 1, \infty$.

Proof. The conjugate exponent $q \in (1, \infty)$. Let $x \in X$. Here is the key idea:

$$|K(x,y)f(y)| = |K(x,y)|^{1/q} \left(|K(x,y)|^{1/p} |f(y)| \right).$$

Apply Hölder's inequality to get

$$\begin{aligned} \int |K(x,y)f(y)| \, dy &\le \left(\int |K(x,y)| \, d\nu(y) \right)^{1/q} \left(\int \left(|K(x,y)|^{1/p} |f(y)| \right)^p \right) d\nu(y)^{1/p} \\ &\le C^{1/q} \left(\int |K(x,y)| |f(g)|^p \, d\nu(y) \right)^{1/p}. \end{aligned}$$

By Tonelli's theorem,

$$\begin{aligned} \int \left[\int |K(x,y)| |f(g)|^p \, d\nu(y) \right] d\mu(x) &= \int \left[\int |K(x,y)| \, d\mu(x) \right] |f(y)|^p |f(y)|^p \, d\nu(y) \\ &\le C \int |f(y)|^p \, d\nu(y) = C \|f\|_{L^p(\nu)}. \end{aligned}$$

Overall, we get

$$\begin{aligned} \int \left[\int |K(x,y)| |f(y)| \, d\nu(y) \right]^p d\mu(x) &\le C^{p/q} \int \int |K(x,y)| |f(y)|^p \, d\nu(y) \, d\mu(x) \\ &\le C^{p/q} C \|f\|_{L^p(\nu)}^p \\ &= C^p \|f\|_{L^p(\nu)}^p. \end{aligned}$$

So $Tf(x)$ is well-defined μ-a.e., and $\|Tf\|_p^p \le \text{LHS} \le C^p \|f\|_p^p$, so $\|T\|_{\mathcal{L}(L^p, L^p)} \le C$. $\square$